

On integrability of discrete variational systems. Octahedron relations

Raphael Boll

Matteo Petrera

Yuri B. Suris

June 4, 2014

Institut für Mathematik, MA 7-2, Technische Universität Berlin,
Straße des 17. Juni 136, 10623 Berlin, Germany

E-mail: boll, petrera, suris@math.tu-berlin.de

Abstract

We elucidate consistency of the so-called corner equations which are elementary building blocks of Euler-Lagrange equations for two-dimensional pluri-Lagrangian problems. We show that their consistency can be derived from the existence of two independent octahedron relations. We give explicit formulas for octahedron relations in terms of corner equations.

Subject areas: mathematical physics, differential equations, field theory

Keywords: discrete integrable systems, Euler-Lagrange equations, variational systems, pluri-Lagrangian systems, multi-dimensional consistency, fractional ideals

1. Introduction

This paper contributes to the Lagrangian theory of discrete integrable systems. This theory emerged recently, following several important developments. Its starting point is the understanding of integrability of discrete hyperbolic systems as their multi-dimensional consistency [BS02, Nij02]. This major breakthrough led to the classification of integrable quad-equations (discrete two-dimensional hyperbolic systems) [ABS03], which turned out to be rather influential. A further conceptual development was initiated by Lobb and Nijhoff [LN09] and deals with the variational (Lagrangian) formulation of discrete multi-dimensionally consistent systems. Their original idea can be summarized as follows: solutions of integrable quad-equations deliver critical points for actions along all possible quad-surfaces in multi-time; the Lagrangian form is closed on solutions.

Solutions of hyperbolic quad-equations do not exhaust critical points for actions along all possible quad-surfaces. In [BPS14], we pushed forward the idea of considering the corresponding Euler-Lagrange equations and general solutions thereof as the proper notion of integrability for discrete variational systems. This is formalized in the following definition:

- Let \mathcal{L} be a discrete 2-form, i.e., a real-valued function of oriented elementary squares

$$\sigma_{ij} = (n, n + e_i, n + e_i + e_j, n + e_j)$$

2. Consistent systems of corner equations

of \mathbb{Z}^m , such that $\mathcal{L}(\sigma_{ij}) = -\mathcal{L}(\sigma_{ji})$. It is assumed to depend on some field $x : \mathbb{Z}^m \rightarrow \mathcal{X}$ assigned to the points of \mathbb{Z}^m (\mathcal{X} being some vector space).

- To an arbitrary oriented quad-surface Σ in \mathbb{Z}^m , there corresponds the *action functional*, which assigns to $x|_{V(\Sigma)}$, i.e., to the fields at the vertices of the surface Σ , the number

$$S_\Sigma := \sum_{\sigma_{ij} \in \Sigma} \mathcal{L}(\sigma_{ij}).$$

- We say that the field $x : V(\Sigma) \rightarrow \mathcal{X}$ is a critical point of S_Σ , if at any interior point $n \in V(\Sigma)$, we have

$$\frac{\partial S_\Sigma}{\partial x(n)} = 0. \quad (1)$$

Equations (1) are called *discrete Euler-Lagrange equations* for the action S_Σ .

- We say that the field $x : \mathbb{Z}^m \rightarrow \mathcal{X}$ solves the *pluri-Lagrangian problem* for the Lagrangian 2-form \mathcal{L} if, for any quad-surface Σ in \mathbb{Z}^m , the restriction $x|_{V(\Sigma)}$ is a critical point of the corresponding action S_Σ .

Discrete Euler-Lagrange equations for the surface Σ of a unit lattice cube are called *corner equations* and are, so to say, elementary particles, of which all possible Euler-Lagrange equations are built of. In general, the system of corner equations for one unit cube is heavily overdetermined, and its consistency is, in our view, synonymous with the integrability of the corresponding pluri-Lagrangian problem. We refer to [BPS14] for details, as well as for some bibliographical and historical remarks concerning this definition.

It is the purpose of the present paper to contribute to a better understanding of algebraic mechanisms behind consistency of the system of corner equations for a class of discrete 2-forms coming from quad-equations of the ABS-list [LN09, BS10]. The corresponding system of corner equations consists of six equations per elementary cube $(x, x_1, x_2, x_3, x_{12}, x_{23}, x_{13}, x_{123})$, each depending on five out of the six variables $x_1, x_2, x_3, x_{12}, x_{23}$ and x_{13} . The system is *consistent* if it has minimal possible rank 2, i.e., if exactly two of these equations are independent. We will demonstrate that one can view consistency of the corner equations as a corollary of the existence of two *octahedron relations*. The latter are multi-affine relations for the six variables $x_1, x_2, x_3, x_{12}, x_{23}$ and x_{13} , satisfied on each solution of corner equations, and, in their turn, having all six corner equations as their corollaries.

2. Consistent systems of corner equations

Definition 2.1 (System of corner equations). For a given discrete 2-form \mathcal{L} , the *system of corner equations* consists of the discrete Euler-Lagrange equations for a surface of an elementary 3D cube.

In the present paper, we consider consistent systems of corner equations having their origin in 3D consistent systems of quad-equations from the ABS-list [ABS03]. Recall [ABS03, LN09] that the corresponding discrete 2-forms are of the following special, three-point shape:

$$\mathcal{L}(\sigma_{ij}) = \mathcal{L}(X, X_i, X_j; \alpha_i, \alpha_j) = L(X, X_i; \alpha_i) - L(X, X_j; \alpha_j) - \Lambda(X_i, X_j; \alpha_i, \alpha_j).$$

2. Consistent systems of corner equations

Setting

$$\frac{\partial L(X, X_i; \alpha_i)}{\partial X} = \psi(X, X_i; \alpha_i), \quad \frac{\partial \Lambda(X_i, X_j; \alpha_i, \alpha_j)}{\partial X_i} = \phi(X_i, X_j; \alpha_i, \alpha_j),$$

we arrive at the following four-leg corner equations for the vertices x_i and x_{ij} , respectively:

$$\psi(X_i, X_{ij}; \alpha_j) - \psi(X_i, X_{ik}; \alpha_k) - \phi(X_i, X_k; \alpha_i, \alpha_k) + \phi(X_i, X_j; \alpha_i, \alpha_j) = 0, \quad (E_i)$$

$$\psi(X_{ij}, X_i; \alpha_j) - \psi(X_{ij}, X_j; \alpha_i) - \phi(X_{ij}, X_{ik}; \alpha_j, \alpha_k) + \phi(X_{ij}, X_{jk}; \alpha_i, \alpha_k) = 0. \quad (E_{ij})$$

Observe that the corner equations for the vertices x and x_{123} are vacuous, and that, moreover, corner equations (E_i) and (E_{ij}) do not involve the fields X and X_{123} .

We recall the relation with the quad-equations: every multi-affine quad-equation from the ABS list,

$$Q(x, x_i, x_j, x_{ij}; \alpha_i, \alpha_j) = 0,$$

admits, after a certain change of variables $x = f(X)$, four equivalent three-leg forms, centered at each of the vertices of an elementary square σ_{ij} , for instance, the one centered at x_i reads:

$$Q = 0 \quad \Leftrightarrow \quad \psi(X_i, X; \alpha_i) - \psi(X_i, X_{ij}; \alpha_j) - \phi(X_i, X_j; \alpha_i, \alpha_j) = 0.$$

Now, each of the four-leg corner equations is a sum of two three-leg equations for two adjacent elementary squares, e.g., equation (E_i) is the difference of two three-leg forms centered at x_i :

$$\psi(X_i, X; \alpha_i) - \psi(X_i, X_{ij}; \alpha_j) - \phi(X_i, X_j; \alpha_i, \alpha_j) = 0, \quad (2)$$

$$\psi(X_i, X; \alpha_i) - \psi(X_i, X_{ik}; \alpha_k) - \phi(X_i, X_k; \alpha_i, \alpha_k) = 0. \quad (3)$$

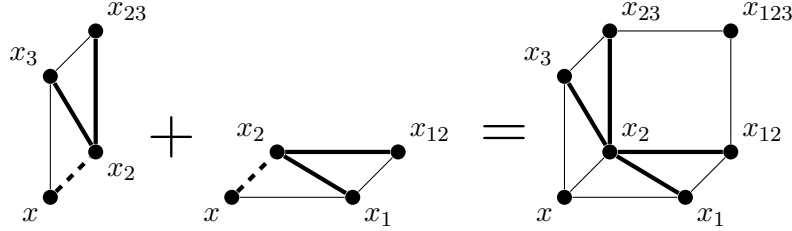


Figure 1: A corner equation for a discrete three-point 2-form can be written as a sum of the three-leg forms of two adjacent quad-equations

Thus, to every system of corner equations (E_i) , (E_{ij}) there corresponds a system of quad-equations:

$$\begin{aligned} Q_{12} &:= Q(x, x_1, x_2, x_{12}; \alpha_1, \alpha_2) = 0, & \bar{Q}_{12} &:= Q(x_3, x_{13}, x_{23}, x_{123}; \alpha_1, \alpha_2) = 0, \\ Q_{23} &:= Q(x, x_2, x_3, x_{23}; \alpha_2, \alpha_3) = 0, & \bar{Q}_{23} &:= Q(x_1, x_{12}, x_{13}, x_{123}; \alpha_2, \alpha_3) = 0, \\ Q_{13} &:= Q(x, x_3, x_1, x_{13}; \alpha_3, \alpha_1) = 0, & \bar{Q}_{13} &:= Q(x_2, x_{23}, x_{13}, x_{123}; \alpha_3, \alpha_1) = 0, \end{aligned} \quad (4)$$

with the following property: every solution of the latter system satisfies the system of corner equations, but not vice versa.

The ABS list of irreducible multi-affine polynomials $Q \in \mathbb{C}[x, x_1, x_2, x_{12}]$ depending on two parameters α_1, α_2 is given in Appendix A. It is known that for every system of quad-equations (4) from the ABS list, there exist two multi-affine quad-equations

$$T(x, x_{12}, x_{23}, x_{13}) = 0 \quad \text{and} \quad \bar{T}(x_1, x_2, x_3, x_{123}) = 0$$

satisfied for every solution of (4) (the so called *tetrahedron equations*).

2. Consistent systems of corner equations

Theorem 2.2. *The system of corner equations (E_i) , (E_{ij}) is equivalent to the following system of six polynomial equations:*

$$\begin{aligned} E_1 &:= \frac{\partial Q_{12}}{\partial x} Q_{13} - \frac{\partial Q_{13}}{\partial x} Q_{12} = 0, & E_{23} &:= \frac{\partial \bar{Q}_{13}}{\partial x_{123}} \bar{Q}_{12} - \frac{\partial \bar{Q}_{12}}{\partial x_{123}} \bar{Q}_{13} = 0, \\ E_2 &:= \frac{\partial Q_{23}}{\partial x} Q_{12} - \frac{\partial Q_{12}}{\partial x} Q_{23} = 0, & E_{13} &:= \frac{\partial \bar{Q}_{12}}{\partial x_{123}} \bar{Q}_{23} - \frac{\partial \bar{Q}_{23}}{\partial x_{123}} \bar{Q}_{12} = 0, \\ E_3 &:= \frac{\partial Q_{13}}{\partial x} Q_{23} - \frac{\partial Q_{23}}{\partial x} Q_{13} = 0, & E_{12} &:= \frac{\partial \bar{Q}_{23}}{\partial x_{123}} \bar{Q}_{13} - \frac{\partial \bar{Q}_{13}}{\partial x_{123}} \bar{Q}_{23} = 0. \end{aligned} \quad (5)$$

We call polynomials E_i , E_{ij} corner polynomials.

Proof. The derivation of the corner equation (E_i) from the three-leg equations (2), (3) can be described as elimination of the variable X between the latter two equations. Obviously, one obtains an equivalent equation by eliminating the variable x between the multi-affine forms $Q_{ij} = 0$ and $Q_{ik} = 0$ of these same equations. \square

We remark that each polynomial E_ℓ ($\ell \in \{1, 2, 3, 12, 23, 13\}$) is of degree 2 with respect to the variable x_ℓ , does not depend on the “opposite” variable, and is of degree one with respect to all other variables.

From now on, we will refer to (5) as to the system of corner equations corresponding to quad-equations (4), or simply as to the system of corner equations. Our main interest is in the following crucial property of such systems.

Definition 2.3. A system of corner equations is called *consistent*, if it has the minimal possible rank 2, i.e., if exactly two of these equations are independent.

In [BPS14], we already gave two different proofs of consistency of the systems of corner equations coming from integrable quad-equations. One of these proofs directly utilized 3D consistency of quad-equations. In the present paper, we provide additional insights in the algebraic structure of the systems of corner equations, which lead to new insights into the nature of their consistency, as well.

We start by establishing certain remarkable relations between corner equations.

Proposition 2.4. *For a system of corner polynomials from (5) the following relations hold:*

$$\frac{\partial E_3}{\partial x_{23}} E_2 - \frac{\partial E_2}{\partial x_{23}} E_3 = Q_{23}^{2,3} E_1, \quad (6)$$

$$\frac{\partial E_3}{\partial x_1} E_2 - \frac{\partial E_2}{\partial x_1} E_3 = Q_{23}^{2,3} E_{23}, \quad (7)$$

and

$$\frac{\partial E_{13}}{\partial x_1} E_{12} - \frac{\partial E_{12}}{\partial x_1} E_{13} = \bar{Q}_{23}^{12,13} E_{23}, \quad (8)$$

$$\frac{\partial E_{13}}{\partial x_{23}} E_{12} - \frac{\partial E_{12}}{\partial x_{23}} E_{13} = \bar{Q}_{23}^{12,13} E_1, \quad (9)$$

where $Q_{23}^{2,3}$ and $\bar{Q}_{23}^{12,13}$ are the biquadratic polynomials

$$Q_{23}^{2,3} = Q_{23} \frac{\partial^2 Q_{23}}{\partial x \partial x_{23}} - \frac{\partial Q_{23}}{\partial x} \frac{\partial Q_{23}}{\partial x_{23}},$$

2. Consistent systems of corner equations

and

$$\bar{Q}_{23}^{12,13} = \bar{Q}_{23} \frac{\partial^2 \bar{Q}_{23}}{\partial x_1 \partial x_{123}} - \frac{\partial \bar{Q}_{23}}{\partial x_1} \frac{\partial \bar{Q}_{23}}{\partial x_{123}}$$

of the quad-equations $Q_{23} = 0$ and $\bar{Q}_{23} = 0$, respectively (superscripts indicate variables on which these biquadratic polynomials depend). Due to the symmetry of the system of quad-equations (4), equations obtained from (6)–(9) by cyclic permutations of indices (123) hold true, as well.

Proof. The proof of (6) is obtained by a direct computation with expressions for E_2, E_3 given in (5). The proof of (7) is substantially more involved. One starts with the observation that the four-leg equation (E_3), as given in (E_i) for $i = 3$, can be alternatively obtained as a sum of the three-leg forms (centered at x_3) of the quad-equation $\bar{Q}_{12}(x_3, x_{13}, x_{23}, x_{123}; \alpha_1, \alpha_2) = 0$ and the tetrahedron equation $\bar{T}(x_1, x_2, x_3, x_{123}) = 0$, respectively:

$$\begin{aligned} \psi(X_3, X_{13}; \alpha_1) - \psi(X_3, X_{23}; \alpha_2) - \phi(X_3, X_{123}; \alpha_1, \alpha_2) &= 0, \\ \phi(X_3, X_1; \alpha_3, \alpha_1) + \phi(X_3, X_2; \alpha_2, \alpha_3) + \phi(X_3, X_{123}; \alpha_1, \alpha_2) &= 0. \end{aligned}$$

Therefore, the polynomial equation $E_3 = 0$ can be alternatively obtained by eliminating the variable x_{123} between the polynomial equations $\bar{Q}_{12} = 0$ and $\bar{T} = 0$. Choosing a suitable normalization of the polynomial \bar{T} (which is defined up to a constant factor), we can assume that

$$E_3 = \frac{\partial \bar{T}}{\partial x_{123}} \bar{Q}_{12} - \frac{\partial \bar{Q}_{12}}{\partial x_{123}} \bar{T}. \quad (10)$$

Analogously,

$$E_2 = \frac{\partial \bar{Q}_{13}}{\partial x_{123}} \tilde{T} - \frac{\partial \tilde{T}}{\partial x_{123}} \bar{Q}_{13}, \quad (11)$$

with $\tilde{T} = \gamma \bar{T}$ being some other normalization of the same tetrahedron equation. Substituting expressions (10), (11) into the left-hand side of (7), one arrives after a straightforward computation, using that all polynomials $\bar{Q}_{12}, \bar{Q}_{13}, \bar{T}$ are multi-affine:

$$\frac{\partial E_3}{\partial x_1} E_2 - \frac{\partial E_2}{\partial x_1} E_3 = \gamma \bar{T}^{2,3} E_{23},$$

with the biquadratic polynomial

$$\bar{T}^{2,3} = \bar{T} \frac{\partial^2 \bar{T}}{\partial x_1 \partial x_{123}} - \frac{\partial \bar{T}}{\partial x_1} \frac{\partial \bar{T}}{\partial x_{123}}.$$

It remains to prove that $\gamma \bar{T}^{2,3} = Q_{23}^{2,3}$. A straightforward computation with the expression (10) for E_3 gives:

$$\frac{\partial \bar{Q}_{12}}{\partial x_{23}} E_3 - \frac{\partial E_3}{\partial x_{23}} \bar{Q}_{12} = \bar{Q}_{12}^{3,13} \bar{T},$$

which can be considered as a formula for the multi-affine polynomial \bar{T} . Computing its biquadratic $\bar{T}^{2,3}$ by a standard Wronskian-type operation eliminating the variables x_1, x_{123} (on which the polynomial $\bar{Q}_{12}^{3,13}$ does not depend), we arrive at

$$\bar{T}^{2,3} = \bar{\beta}^{-1} Q_{23}^{2,3}, \quad \text{where} \quad \bar{\beta} := \frac{\bar{Q}_{12}^{3,13}}{Q_{13}^{3,13}}.$$

3. Consistency of corner equations and octahedron relations

Similarly, a computation with the expression (11) for E_2 gives:

$$\frac{\partial E_2}{\partial x_{23}} \bar{Q}_{13} - \frac{\partial \bar{Q}_{13}}{\partial x_{23}} E_2 = \bar{Q}_{13}^{2,12} \tilde{T},$$

and then we find an expression for the biquadratic $\tilde{T}^{2,3}$ of the multi-affine polynomial \tilde{T} :

$$\tilde{T}^{2,3} = \tilde{\beta} Q_{23}^{2,3} \quad \text{where} \quad \tilde{\beta} := \frac{Q_{12}^{2,12}}{Q_{13}^{2,12}}.$$

But, according to [ABS03, Proposition 5], we have:

$$\frac{Q_{12}^{2,12}}{Q_{13}^{2,12}} = \frac{\bar{Q}_{12}^{3,13}}{Q_{13}^{3,13}}, \quad \text{i.e.,} \quad \bar{\beta} = \tilde{\beta} =: \beta.$$

Therefore, $\tilde{T}^{2,3} = \beta^2 \bar{T}^{2,3}$, so that $\tilde{T} = \beta \bar{T}$ or $\tilde{T} = -\beta \bar{T}$. A case-by-case inspection of the ABS list shows that $\tilde{T} = \beta \bar{T}$, so $\beta = \gamma$, and therefore $Q_{23}^{2,3} = \gamma \bar{T}^{2,3}$, which finishes the proof of equation (7). The proofs of equations (8) and (9) are analogous to the proofs of equations (6) and (7), respectively. \square

We now recall the definition of fractional ideals, cf. [Bou72, section VII.1], adapted to our situation:

Definition 2.5 (Fractional ideal). Set $\mathcal{R} := \mathbb{C}[x_1, x_2, x_3, x_{12}, x_{23}, x_{13}]$ and $r \in \mathcal{R} \setminus \{0\}$. Then a *fractional ideal* \mathcal{I} with the denominator r is an \mathcal{R} -submodule of the field of fractions of \mathcal{R} such that $r\mathcal{I} \subset \mathcal{R}$. For three polynomials $p_1, p_2, r \in \mathcal{R} \setminus \{0\}$, the *fractional ideal generated by p_1 and p_2 with the denominator r* is denoted by $\langle p_1, p_2 \rangle_r$ and consists of all polynomials p representable as $rp = r_1 p_1 + r_2 p_2$ with some $r_1, r_2 \in \mathcal{R}$. We say that the polynomial $p \in \mathcal{R}$ is in a fractional ideal generated by two polynomials $p_1, p_2 \in \mathcal{R}$ if $p \in \langle p_1, p_2 \rangle_r$ for some $r \in \mathcal{R}$.

With the help of this definition, we can re-phrase Proposition 2.4 by saying that E_2, E_3, E_{23} and E_{13} are in a fractional ideal generated by E_1 and E_{23} . This easily yields the following more general statement which can be considered as a description of consistency of corner equations in terms of fractional ideals:

Theorem 2.6. *Any two of the corner polynomials from the system (5) generate a fractional ideal which contains the remaining four corner polynomials. Therefore, the rank of the system (5) is 2.*

3. Consistency of corner equations and octahedron relations

In view of Theorem 2.6, it is natural to inquire whether one can find two relations which would be either simpler or more symmetric than the corner equations themselves but which would still contain the whole information of the system (5), for instance in the sense that all corner equations are in a fractional ideal generated by those two relations. We will see that one can find such two relations in the multi-affine class.

Definition 3.1 (Octahedron relation). A relation $R = 0$, where $R \in \mathbb{C}[x_1, x_2, x_3, x_{12}, x_{23}, x_{13}]$ is a multi-affine polynomial, is an *octahedron relation* of a system of corner equations (5) if R belongs to a fractional ideal generated by (any two of) the corner polynomials E_i, E_{ij} .

4. First octahedron relation

Immediately from this definition there follows:

Theorem 3.2. *Suppose the system of corner equations (5) admits two octahedron relations $R_1 = 0$ and $R_2 = 0$, where R_1, R_2 are two linearly independent irreducible polynomials. Then all corner polynomials are in a fractional ideal generated by R_1 and R_2 , which is equivalent to the consistency of the system (5).*

Remark. Actually, one can give a more precise formulation: the corner polynomials E_i, E_{jk} divide the polynomials

$$B_i := \frac{\partial R_2}{\partial x_{jk}} R_1 - \frac{\partial R_1}{\partial x_{jk}} R_2, \quad \text{resp.} \quad B_{jk} := \frac{\partial R_2}{\partial x_i} R_1 - \frac{\partial R_1}{\partial x_i} R_2.$$

To show this for $E_1 = 0$, say (since for all other corner polynomials the proof is analogous), we observe that the polynomial B_1 is a non-zero polynomial, independent of x_{23} and of degree 2 with respect to all other variables x_1, x_2, x_3, x_{12} and x_{13} . By definition, B_1 is in a fractional ideal generated by E_1 and E_{23} . The coefficient by E_{23} in the corresponding representation $rB = p_1 E_1 + p_{23} E_{23}$ must vanish, since both B_1 and E_1 do not depend on x_{23} . Therefore, $B_1 = 0$ as soon as $E_1 = 0$, hence B_1 is divisible by E_1 .

One of the main results of this paper is the following.

Theorem 3.3 (Consistency of corner equations). *A system of corner equations (5) coming from any of the ABS quad-equations admits two independent octahedron relations.*

We will prove this by an explicit construction of the octahedron relations.

4. First octahedron relation

The first octahedron relation can be found in the following way:

Theorem 4.1. *Equation $R_1 = 0$ with*

$$R_1 := \frac{1}{2} \left(\frac{\partial E_1}{\partial x_1} + \frac{\partial E_{23}}{\partial x_{23}} \right)$$

is an octahedron relation of the system (5). Analogous octahedron relations are obtained through cyclic permutations of indices (123):

$$R_2 := \frac{1}{2} \left(\frac{\partial E_2}{\partial x_2} + \frac{\partial E_{13}}{\partial x_{13}} \right), \quad R_3 := \frac{1}{2} \left(\frac{\partial E_3}{\partial x_3} + \frac{\partial E_{12}}{\partial x_{12}} \right).$$

Proof. Since polynomials E_1 and E_{23} are of degree 2 with respect to the variables x_1 , resp. x_{23} , and of degree 1 with respect to all other variables, it follows that polynomial R_1 is multi-affine. Using equations (6) and (7), we find:

$$\begin{aligned} \frac{\partial^2 E_3}{\partial x_1 \partial x_{23}} E_2 - \frac{\partial^2 E_2}{\partial x_1 \partial x_{23}} E_3 &= \frac{1}{2} \left(\frac{\partial}{\partial x_1} \left(\frac{\partial E_3}{\partial x_{23}} E_2 - \frac{\partial E_2}{\partial x_{23}} E_3 \right) + \frac{\partial}{\partial x_{23}} \left(\frac{\partial E_3}{\partial x_1} E_2 - \frac{\partial E_2}{\partial x_1} E_3 \right) \right) \\ &= \frac{1}{2} Q_{23}^{2,3} \left(\frac{\partial E_1}{\partial x_1} + \frac{\partial E_{23}}{\partial x_{23}} \right) = Q_{23}^{2,3} R_1. \end{aligned}$$

Therefore, R_1 is in a fractional ideal generated by E_2 and E_3 (and therefore in a fractional ideal generated by any other two corner equations from (5)). \square

4. First octahedron relation

The following proposition is based on straightforward case-by-case computations for all ABS equations:

Proposition 4.2.

- In the case coming from the quad-equation A_2 , the following equation is satisfied:

$$\frac{\sin(\alpha_2 - \alpha_3)}{\sin(\alpha_1)} R_1 + \frac{\sin(\alpha_3 - \alpha_1)}{\sin(\alpha_2)} R_2 + \frac{\sin(\alpha_1 - \alpha_2)}{\sin(\alpha_3)} R_3 = 0.$$

- In the case coming from the quad-equation Q_4 , the following equation is satisfied:

$$\text{sn}(\alpha_2 - \alpha_3) R_1 + \text{sn}(\alpha_3 - \alpha_1) R_2 + \text{sn}(\alpha_1 - \alpha_2) R_3 = 0.$$

- In these two cases, any two of the polynomials R_1, R_2, R_3 are linearly independent.
- In all other cases we have:

$$\frac{R_1}{f_1} = \frac{R_2}{f_2} = \frac{R_3}{f_3} =: \Omega_1,$$

with the constant factors

$$f_1 := \frac{\partial^2 Q_{12}}{\partial x \partial x_1} = -\frac{\partial^2 Q_{13}}{\partial x \partial x_1}, \quad f_2 := \frac{\partial^2 Q_{23}}{\partial x \partial x_1} = -\frac{\partial^2 Q_{12}}{\partial x \partial x_1}, \quad f_3 := \frac{\partial^2 Q_{13}}{\partial x \partial x_3} = -\frac{\partial^2 Q_{23}}{\partial x \partial x_3}.$$

Thus, in the cases coming from A_2 and Q_4 , all corner equations from the system (5) are in a fractional ideal generated by R_1 and R_2 . In all other cases, one has just one octahedron relation $\Omega_1 = 0$, symmetric with respect to cyclic permutations of indices (123), and one has to look for further octahedron relations. A list of the polynomials R_1 , resp. Ω_1 , can be found in Appendix B.

Before we turn to the problem of finding the second independent octahedron relation for equations not coming from case A_2 and Q_4 , we mention another expression for the octahedron relation $\Omega_1 = 0$.

Proposition 4.3. *In the cases coming from the quad-equations $Q_1^\delta, Q_2, Q_3^\delta$ and A_1^δ the octahedron relation $\Omega_1 = 0$ can be written as*

$$\Omega_1 = Q_{12} + Q_{23} + Q_{13} + T = 0,$$

where $T = T(x, x_{12}, x_{23}, x_{13}) = 0$ is the tetrahedron equation. In the cases coming from the quad-equations H_1, H_2 and H_3^δ the octahedron relation $\Omega_1 = 0$ can be written as

$$\Omega_1 = Q_{12} + Q_{23} + Q_{13} = 0.$$

Proof. A straightforward computation gives:

$$\Omega_1 = \frac{1}{2f_1} \left(\frac{\partial E_1}{\partial x_1} + \frac{\partial E_{23}}{\partial x_{23}} \right) = Q_{13} + Q_{12} + G_1,$$

where

$$G_1 = \frac{1}{2f_1} \left(\frac{\partial^2 Q_{13}}{\partial x \partial x_1} Q_{12} - \frac{\partial^2 Q_{12}}{\partial x \partial x_1} Q_{13} + \frac{\partial Q_{12}}{\partial x} \frac{\partial Q_{13}}{\partial x_1} - \frac{\partial Q_{13}}{\partial x} \frac{\partial Q_{12}}{\partial x_1} \right) + \frac{1}{2f_1} \left(\frac{\partial^2 \bar{Q}_{13}}{\partial x_{23} \partial x_{123}} \bar{Q}_{12} - \frac{\partial^2 \bar{Q}_{12}}{\partial x_{23} \partial x_{123}} \bar{Q}_{13} + \frac{\partial \bar{Q}_{13}}{\partial x_{123}} \frac{\partial \bar{Q}_{12}}{\partial x_{23}} - \frac{\partial \bar{Q}_{12}}{\partial x_{123}} \frac{\partial \bar{Q}_{13}}{\partial x_{23}} \right).$$

5. Second octahedron relation

Using the fact that all polynomials Q_{ij} , \bar{Q}_{ij} are multi-affine, we immediately compute that

$$\frac{\partial G_1}{\partial x_1} = 0, \quad \frac{\partial G_1}{\partial x_{123}} = 0.$$

Therefore, $G_1 = G_1(x, x_2, x_3, x_{12}, x_{23}, x_{13})$ is multi-affine and independent of x_1 and x_{123} . Analogously,

$$\Omega_1 = \frac{1}{2f_2} \left(\frac{\partial E_2}{\partial x_2} + \frac{\partial E_{13}}{\partial x_{13}} \right) = Q_{12} + Q_{23} + G_2(x, x_3, x_1, x_{12}, x_{23}, x_{13}),$$

where $G_2(x, x_3, x_1, x_{12}, x_{23}, x_{13})$ is multi-affine and independent of x_2 and x_{123} , and

$$\Omega_1 = \frac{1}{2f_3} \left(\frac{\partial E_3}{\partial x_3} + \frac{\partial E_{12}}{\partial x_{12}} \right) = Q_{23} + Q_{13} + G_3(x, x_1, x_2, x_{12}, x_{23}, x_{13}),$$

where $G_3(x, x_1, x_2, x_{12}, x_{23}, x_{13})$ is multi-affine and independent of x_3 and x_{123} . Thus,

$$Q_{13} + Q_{12} + G_1 = Q_{12} + Q_{23} + G_2 = Q_{23} + Q_{13} + G_3.$$

Set

$$T := G_1 - Q_{23} = G_2 - Q_{13} = G_3 - Q_{12}.$$

The first expression for T shows that it is independent of x_1 , the second shows that it is independent of x_2 , and the third shows that it is independent of x_3 . Therefore,

$$\Omega_1 = Q_{12} + Q_{23} + Q_{13} + T(x, x_{12}, x_{23}, x_{13}),$$

where $T(x, x_{12}, x_{23}, x_{13})$ is multi-affine and independent of x_1, x_2, x_3 and x_{123} . One shows by a simple case-by-case that in the cases coming from $Q_1^\delta, Q_2, Q_3^\delta$ and A_1^δ , the polynomial T defines the tetrahedron relation $T = 0$, while in the cases coming from H_1, H_2 and H_3^δ we have $T \equiv 0$. \square

Remark. Of course, in the cases coming from $Q_1^\delta, Q_2, Q_3^\delta$ and A_1^δ one can write the octahedron relation $\Omega_1 = 0$ as

$$\Omega_1 = \bar{Q}_{12} + \bar{Q}_{23} + \bar{Q}_{13} + \bar{T} = 0,$$

where the polynomial $\bar{T} = \bar{T}(x_1, x_2, x_3, x_{123})$ defines the tetrahedron relation $\bar{T} = 0$, while in the cases coming from H_1, H_2 and H_3^δ we have

$$\Omega_1 = \bar{Q}_{12} + \bar{Q}_{23} + \bar{Q}_{13} = 0.$$

5. Second octahedron relation

Turning to the problem of finding the second octahedron relation in the cases different from A_2 and Q_4 , we have the following results.

Theorem 5.1. *The polynomials*

$$P_1 := E_1 - x_1 R_1 \quad \text{and} \quad P_{23} := E_{23} - x_{23} R_1$$

5. Second octahedron relation

satisfy the following identities:

$$\frac{\partial R_1}{\partial x_{23}} P_1 - \frac{\partial P_1}{\partial x_{23}} R_1 = \frac{\partial R_1}{\partial x_{23}} E_1, \quad \frac{\partial R_1}{\partial x_1} P_1 - \frac{\partial P_1}{\partial x_1} R_1 = \frac{\partial R_1}{\partial x_{23}} E_{23}, \quad (12)$$

and, similarly,

$$\frac{\partial R_1}{\partial x_{23}} P_{23} - \frac{\partial P_{23}}{\partial x_{23}} R_1 = \frac{\partial R_1}{\partial x_1} E_1, \quad \frac{\partial R_1}{\partial x_1} P_{23} - \frac{\partial P_{23}}{\partial x_1} R_1 = \frac{\partial R_1}{\partial x_1} E_{13}. \quad (13)$$

Thus, $P_1 = 0$ and $P_{23} = 0$ are octahedron relations of the system (5). One find further octahedron relations under cyclic permutations of indices (123).

Proof. We prove relations (12) for P_1 , the proof of relations (13) for P_{23} being similar. The first of the relations in (12) is trivial:

$$\frac{\partial R_1}{\partial x_{23}} P_1 - \frac{\partial P_1}{\partial x_{23}} R_1 = \frac{\partial R_1}{\partial x_{23}} (E_1 - x_1 R_1) + x_1 \frac{\partial R_1}{\partial x_{23}} R_1 = \frac{\partial R_1}{\partial x_{23}} E_1.$$

As for the second relation in (12), we compute:

$$\begin{aligned} \frac{\partial R_1}{\partial x_1} P_1 - \frac{\partial P_1}{\partial x_1} R_1 - \frac{\partial R_1}{\partial x_{23}} E_{23} &= \frac{1}{2} \frac{\partial^2 E_1}{\partial x_1^2} \left(E_1 - \frac{1}{2} \left(\frac{\partial E_1}{\partial x_1} + \frac{\partial E_{23}}{\partial x_{23}} \right) \right) \\ &\quad - \frac{1}{2} \left(\frac{\partial E_1}{\partial x_1} - \frac{1}{2} \left(\frac{\partial E_1}{\partial x_1} + \frac{\partial E_{23}}{\partial x_{23}} \right) - \frac{1}{2} x_1 \frac{\partial^2 E_1}{\partial x_1^2} \right) \left(\frac{\partial E_1}{\partial x_1} + \frac{\partial E_{23}}{\partial x_{23}} \right) - \frac{1}{2} \frac{\partial^2 E_{23}}{\partial x_{23}^2} E_{23} \\ &= \frac{1}{2} \left(E_1 \frac{\partial^2 E_1}{\partial x_1^2} - \frac{1}{2} \left(\frac{\partial E_1}{\partial x_1} \right)^2 - E_{23} \frac{\partial^2 E_{23}}{\partial x_{23}^2} + \frac{1}{2} \left(\frac{\partial E_{23}}{\partial x_{23}} \right)^2 \right). \end{aligned}$$

This vanishes due to following Lemma 5.2. \square

Lemma 5.2. *The discriminants of the opposite corner polynomials with respect to their central points coincide:*

$$E_1 \frac{\partial^2 E_1}{\partial x_1^2} - \frac{1}{2} \left(\frac{\partial E_1}{\partial x_1} \right)^2 = E_{23} \frac{\partial^2 E_{23}}{\partial x_{23}^2} - \frac{1}{2} \left(\frac{\partial E_{23}}{\partial x_{23}} \right)^2.$$

Proof. We have

$$\begin{aligned} E_1 \frac{\partial^2 E_1}{\partial x_1^2} - \frac{1}{2} \left(\frac{\partial E_1}{\partial x_1} \right)^2 &= 2 \left(\frac{\partial Q_{12}}{\partial x} Q_{13} - \frac{\partial Q_{13}}{\partial x} Q_{12} \right) \left(\frac{\partial^2 Q_{12}}{\partial x \partial x_1} \frac{\partial Q_{13}}{\partial x_1} - \frac{\partial^2 Q_{13}}{\partial x \partial x_1} \frac{\partial Q_{12}}{\partial x_1} \right) \\ &\quad - \frac{1}{2} \left(\frac{\partial^2 Q_{12}}{\partial x \partial x_1} Q_{13} + \frac{\partial Q_{12}}{\partial x} \frac{\partial Q_{13}}{\partial x_1} - \frac{\partial^2 Q_{13}}{\partial x \partial x_1} Q_{12} - \frac{\partial Q_{13}}{\partial x} \frac{\partial Q_{12}}{\partial x_1} \right)^2 \\ &= \frac{\partial Q_{13}}{\partial x_1} Q_{13} \frac{\partial^2 Q_{12}}{\partial x \partial x_1} \frac{\partial Q_{12}}{\partial x} - 2 \frac{\partial^2 Q_{13}}{\partial x \partial x_1} Q_{13} \frac{\partial Q_{12}}{\partial x} \frac{\partial Q_{12}}{\partial x_1} - 2 \frac{\partial Q_{13}}{\partial x} \frac{\partial Q_{13}}{\partial x_1} \frac{\partial^2 Q_{12}}{\partial x \partial x_1} Q_{12} \\ &\quad + \frac{\partial^2 Q_{13}}{\partial x \partial x_1} \frac{\partial Q_{13}}{\partial x} \frac{\partial Q_{12}}{\partial x_1} Q_{12} - \frac{1}{2} \left(Q_{13} \frac{\partial^2 Q_{12}}{\partial x \partial x_1} \right)^2 - \frac{1}{2} \left(\frac{\partial Q_{13}}{\partial x_1} \frac{\partial Q_{12}}{\partial x} \right)^2 - \frac{1}{2} \left(\frac{\partial^2 Q_{13}}{\partial x \partial x_1} Q_{12} \right)^2 \\ &\quad - \frac{1}{2} \left(\frac{\partial Q_{13}}{\partial x} \frac{\partial Q_{12}}{\partial x_1} \right)^2 + \frac{\partial^2 Q_{13}}{\partial x \partial x_1} Q_{13} \frac{\partial^2 Q_{12}}{\partial x \partial x_1} Q_{12} + \frac{\partial Q_{13}}{\partial x} Q_{13} \frac{\partial^2 Q_{12}}{\partial x \partial x_1} \frac{\partial Q_{12}}{\partial x_1} \\ &\quad + \frac{\partial^2 Q_{13}}{\partial x \partial x_1} \frac{\partial Q_{13}}{\partial x} \frac{\partial Q_{12}}{\partial x} Q_{12} + \frac{\partial Q_{13}}{\partial x} \frac{\partial Q_{13}}{\partial x_1} \frac{\partial Q_{12}}{\partial x} \frac{\partial Q_{12}}{\partial x_1}. \end{aligned}$$

5. Second octahedron relation

This expression is manifestly symmetric with respect to the permutation $x \leftrightarrow x_1$. Therefore,

$$E_1 \frac{\partial^2 E_1}{\partial x_1^2} - \frac{1}{2} \left(\frac{\partial E_1}{\partial x_1} \right)^2 = F_1 \frac{\partial^2 F_1}{\partial x^2} - \frac{1}{2} \left(\frac{\partial F_1}{\partial x} \right)^2,$$

where

$$F_1 := \frac{\partial Q_{12}}{\partial x_1} Q_{13} - \frac{\partial Q_{13}}{\partial x_1} Q_{12}.$$

It remains to prove that

$$F_1 \frac{\partial^2 F_1}{\partial x^2} - \frac{1}{2} \left(\frac{\partial F_1}{\partial x} \right)^2 = E_{23} \frac{\partial^2 E_{23}}{\partial x_{23}^2} - \frac{1}{2} \left(\frac{\partial E_{23}}{\partial x_{23}} \right)^2. \quad (14)$$

For this aim, we proceed similarly to the proof of Proposition 2.4. One can represent the equation $F_1 = 0$ as the result of eliminating the variable x_{23} between $Q_{23} = 0$ and the suitably normalized tetrahedron equation $T = T(x, x_{12}, x_{23}, x_{13}) = 0$:

$$F_1 = \frac{\partial Q_{23}}{\partial x_{23}} T - \frac{\partial T}{\partial x_{23}} Q_{23}.$$

On the other hand, we have a similar representation of the corner equation $E_{23} = 0$ as the result of eliminating the variable x between $Q_{23} = 0$ and the possibly differently normalized tetrahedron equation $T = 0$:

$$E_{23} = \beta \left(\frac{\partial Q_{23}}{\partial x} T - \frac{\partial T}{\partial x} Q_{23} \right).$$

A direct case-by-case check shows that for all ABS equations we have $\beta^2 = 1$. This yields (14) by a straightforward computation. \square

The following proposition is proved by case-by-case computations.

Proposition 5.3. *For systems of corner equations (5) coming from Q_1^δ , Q_2 , Q_3^δ , H_1 , H_2 , H_3^δ or A_1^δ , the octahedron relation $\Omega_2 = 0$ with*

$$\Omega_2 := \frac{1}{g_1} (P_1 - P_{23})$$

is symmetric with respect to the cyclic permutation (123) of indices. Here, g_1 is a constant factor given by

$$g_1 := \begin{cases} f_1(\alpha_2 + \alpha_3 - \alpha_1) & \text{if (5) comes from } Q_1^\delta \text{ or } Q_2, \\ 2f_1 e^{i(\alpha_1 + \alpha_2 + \alpha_3)} \sin\left(\frac{1}{2}(\alpha_2 + \alpha_3 - \alpha_1)\right) & \text{if (5) comes from } Q_3^\delta, \\ f_1 & \text{if (5) comes from } H_1, H_2 \text{ or } A_1^\delta, \\ 1 & \text{if (5) comes from } H_3^\delta. \end{cases}$$

The list of the polynomials Ω_2 can be found in Appendix B.

From Theorem 5.1 and Proposition 5.3 we get the following corollary:

6. Comparison with previously known results

Corollary 5.4. *Corner equations (5) coming from Q_1^δ , Q_2 , Q_3^δ , H_1 , H_2 , H_3^δ or A_1^δ , are expressed through the octahedron equations $\Omega_1 = 0$ and $\Omega_2 = 0$ as follows:*

$$\begin{aligned}\frac{\partial \Omega_1}{\partial x_{23}} \Omega_2 - \frac{\partial \Omega_2}{\partial x_{23}} \Omega_1 &= g_1 \left(\frac{\partial \Omega_1}{\partial x_{23}} - \frac{\partial \Omega_1}{\partial x_1} \right) E_1, \\ \frac{\partial \Omega_1}{\partial x_1} \Omega_2 - \frac{\partial \Omega_2}{\partial x_1} \Omega_1 &= g_1 \left(\frac{\partial \Omega_1}{\partial x_{23}} - \frac{\partial \Omega_1}{\partial x_1} \right) E_{23}.\end{aligned}$$

Analogous formulas hold under cyclic permutations of indices (123).

These formulas give a concrete realization of the remark after Theorem 3.2 for the polynomials Ω_1, Ω_2 .

6. Comparison with previously known results

In our recent paper [BPS14] we already gave a list of single octahedron relations for several systems of corner equations, namely, in the cases coming from Q_1^δ , Q_3^0 , H_1 , H_2 and H_3^δ .

In the cases coming from H_1 , H_2 and H_3^δ , they are equivalent to the relations $\Omega_1 = 0$ found in the present paper.

In the case coming from Q_1^0 , we presented in [BPS14] the octahedron relation

$$\frac{(x_{12} - x_1)(x_{23} - x_2)(x_{13} - x_3)}{(x_{12} - x_2)(x_{23} - x_3)(x_{13} - x_1)} = 1 \quad (15)$$

(equation (χ_2) in the classification of [ABS12]), which turns out to be equivalent to $\Omega_2 = 0$ found in the present paper.

In the case coming from Q_1^1 we presented in [BPS14] the octahedron relation resembling equation (15):

$$\frac{(x_{12} - x_1 + \alpha_2)(x_{23} - x_2 + \alpha_3)(x_{13} - x_3 + \alpha_1)}{(x_{12} - x_2 + \alpha_1)(x_{23} - x_3 + \alpha_2)(x_{13} - x_1 + \alpha_3)} = 1.$$

This is equivalent to $P^+ = 0$, where

$$\begin{aligned}P^+ &= (x_{12} - x_2 + \alpha_1)(x_{23} - x_3 + \alpha_2)(x_{13} - x_1 + \alpha_3) \\ &\quad - (x_{12} - x_1 + \alpha_2)(x_{23} - x_2 + \alpha_3)(x_{13} - x_3 + \alpha_1).\end{aligned}$$

Another octahedron relation $P^- = 0$ could be obtained by inverting the signs of all parameters α_i (this operation leaves corner equations invariant):

$$\begin{aligned}P^- &= (x_{12} - x_2 - \alpha_1)(x_{23} - x_3 - \alpha_2)(x_{13} - x_1 - \alpha_3) \\ &\quad - (x_{12} - x_1 - \alpha_2)(x_{23} - x_2 - \alpha_3)(x_{13} - x_3 - \alpha_1).\end{aligned}$$

It turns out that in this case we have the following relations:

$$\Omega_1 = \frac{1}{2}(P^+ + P^-), \quad \Omega_2 = \frac{1}{2}(P^+ - P^-).$$

In the case coming from Q_3^0 we gave an octahedron relation equivalent to $P^+ = 0$, where

$$\begin{aligned}P^+ &= \left(e^{\frac{i}{2}(\alpha_1 - \alpha_2 - \alpha_3)} x_1 + e^{-\frac{i}{2}(\alpha_1 - \alpha_2 - \alpha_3)} x_{23} \right) (x_2 x_{13} - x_3 x_{12}) \\ &\quad + \left(e^{\frac{i}{2}(\alpha_2 - \alpha_3 - \alpha_1)} x_2 + e^{-\frac{i}{2}(\alpha_2 - \alpha_3 - \alpha_1)} x_{13} \right) (x_3 x_{12} - x_1 x_{23}) \\ &\quad + \left(e^{\frac{i}{2}(\alpha_3 - \alpha_1 - \alpha_2)} x_3 + e^{-\frac{i}{2}(\alpha_3 - \alpha_1 - \alpha_2)} x_{12} \right) (x_1 x_{23} - x_2 x_{13}).\end{aligned}$$

6. Comparison with previously known results

Again, another octahedron relation $P^- = 0$ could be obtained by inverting the signs of all parameters α_i :

$$\begin{aligned} P^- = & - \left(e^{-\frac{i}{2}(\alpha_1 - \alpha_2 - \alpha_3)} x_1 + e^{\frac{i}{2}(\alpha_1 - \alpha_2 - \alpha_3)} x_{23} \right) (x_2 x_{13} - x_3 x_{12}) \\ & - \left(e^{-\frac{i}{2}(\alpha_2 - \alpha_3 - \alpha_1)} x_2 + e^{\frac{i}{2}(\alpha_2 - \alpha_3 - \alpha_1)} x_{13} \right) (x_3 x_{12} - x_1 x_{23}) \\ & - \left(e^{-\frac{i}{2}(\alpha_3 - \alpha_1 - \alpha_2)} x_3 + e^{\frac{i}{2}(\alpha_3 - \alpha_1 - \alpha_2)} x_{12} \right) (x_1 x_{23} - x_2 x_{13}). \end{aligned}$$

It turns out that these polynomials P^+ , P^- are related to Ω_1 , Ω_2 found in the present paper through the following formulas:

$$\Omega_2 = \frac{1}{2}(P^+ - P^-), \quad (2x_1 x_{23} - x_2 x_{13} - x_3 x_{13}) i \Omega_1 = p^+ P^+ + p^- P^-,$$

where

$$\begin{aligned} p^+ = & -e^{-\frac{i}{2}(\alpha_1 - \alpha_2 - \alpha_3)} x_1 + \frac{1}{2} e^{-\frac{i}{2}(\alpha_2 - \alpha_3 - \alpha_1)} x_2 + \frac{1}{2} e^{-\frac{i}{2}(\alpha_3 - \alpha_1 - \alpha_2)} x_3 \\ & + e^{\frac{i}{2}(\alpha_1 - \alpha_2 - \alpha_3)} x_{23} - \frac{1}{2} e^{\frac{i}{2}(\alpha_2 - \alpha_3 - \alpha_1)} x_{13} - \frac{1}{2} e^{\frac{i}{2}(\alpha_3 - \alpha_1 - \alpha_2)} x_{12} \end{aligned}$$

and

$$\begin{aligned} p^- = & e^{\frac{i}{2}(\alpha_1 - \alpha_2 - \alpha_3)} x_1 - \frac{1}{2} e^{\frac{i}{2}(\alpha_2 - \alpha_3 - \alpha_1)} x_2 - \frac{1}{2} e^{\frac{i}{2}(\alpha_3 - \alpha_1 - \alpha_2)} x_3 \\ & - e^{-\frac{i}{2}(\alpha_1 - \alpha_2 - \alpha_3)} x_{23} + \frac{1}{2} e^{-\frac{i}{2}(\alpha_2 - \alpha_3 - \alpha_1)} x_{13} + \frac{1}{2} e^{-\frac{i}{2}(\alpha_3 - \alpha_1 - \alpha_2)} x_{12}. \end{aligned}$$

This shows explicitly that P^+ and P^- are in a fractional generated by Ω_1 and Ω_2 .

In [BPS14] we also discussed the system of corner equations

$$\frac{\alpha_j x_i - x_{ij}}{x_i} \cdot \frac{\alpha_i x_i - \alpha_j x_j}{\alpha_j x_i - \alpha_i x_j} \cdot \frac{x_i}{\alpha_k x_i - x_{ik}} \cdot \frac{\alpha_k x_i - \alpha_i x_k}{\alpha_i x_i - \alpha_k x_k} = 1, \quad (E_i)$$

$$\frac{\alpha_j x_i - x_{ij}}{x_i} \cdot \frac{\alpha_j x_{ij} - \alpha_k x_{ik}}{\alpha_k x_{ij} - \alpha_j x_{ik}} \cdot \frac{x_j}{\alpha_i x_j - x_{ij}} \cdot \frac{\alpha_k x_{ij} - \alpha_i x_{jk}}{\alpha_i x_{ij} - \alpha_k x_{jk}} = 1 \quad (E_{ij})$$

(where (i, j, k) is a permutation of $(1, 2, 3)$), which cannot be derived from an integrable system of quad-equations. We gave the octahedron relation $P^+ = 0$, with

$$P^+ = x_1 x_2 (\alpha_1 x_{13} - \alpha_2 x_{23}) + x_2 x_3 (\alpha_2 x_{12} - \alpha_3 x_{13}) + x_3 x_1 (\alpha_3 x_{23} - \alpha_1 x_{12}) = 0.$$

This relation is of the type (χ_4) in the classification from [ABS12]. Another octahedron relation $P^- = 0$, which can also be obtained from the corresponding octahedron relation in the case coming from Q_3^0 using the limiting procedure described in [BPS14], reads as follows:

$$P^- = (\alpha_2 x_1 - x_{12})(\alpha_3 x_2 - x_{23})(\alpha_1 x_3 - x_{13}) - (\alpha_1 x_2 - x_{12})(\alpha_2 x_3 - x_{23})(\alpha_3 x_1 - x_{13}).$$

It turns out that Theorem 4.1 is perfectly applicable in this case, leading to a permutationally symmetric octahedron relation $\Omega_1 = 0$ with

$$\begin{aligned} \Omega_1 = & \frac{1}{2} \left(\frac{\partial E_1}{\partial x_1} + \frac{\partial E_{23}}{\partial x_{23}} \right) \\ = & \alpha_1 \alpha_2 (x_1 x_{13} - x_2 x_{23}) + \alpha_2 \alpha_3 (x_2 x_{12} - x_3 x_{13}) + \alpha_3 \alpha_1 (x_3 x_{23} - x_1 x_{12}) \\ & + \alpha_1^2 (\alpha_3 x_2 - \alpha_2 x_3) + \alpha_2^2 (\alpha_1 x_3 - \alpha_3 x_1) + \alpha_3^2 (\alpha_2 x_1 - \alpha_1 x_2). \end{aligned}$$

7. Conclusion

Polynomial Ω_1 lies in a fractional ideal generated by P^+ , P^- , as the following formulas show:

$$(2x_1x_{23} - x_2x_{13} - x_3x_{12})\Omega_1 = p^+P^+ + p^-P^-,$$

where

$$\begin{aligned} p^+ &= 2\alpha_2\alpha_3x_1 - \alpha_3\alpha_1x_2 - \alpha_1\alpha_2x_3 + 2\alpha_1x_{23} - \alpha_2x_{13} - \alpha_3x_{12}, \\ p^- &= 2\alpha_1x_1 - \alpha_2x_2 - \alpha_3x_3. \end{aligned}$$

Constructions of Theorem 5.1 and Proposition 5.3 are also applicable in this case and lead to further octahedron relations, which however are not permutationally symmetric (but, of course, lie in a fractional ideal generated by P^+ and P^-). For instance, polynomial from Proposition 5.3 satisfies

$$P_1 - P_{23} = (\alpha_1\alpha_3 - \alpha_2)P^+ + \alpha_2P^-.$$

7. Conclusion

We showed that every system of corner equations generated by a discrete 2-form corresponding to quad-equations from the ABS list can be encoded in a system of two octahedron relations: all corner equations follow from and therefore are satisfied by virtue of two octahedron equations. On the other hand, we found explicit and general formulae allowing us to express the octahedron relations in terms of the corner equations. Therefore, the system of corner equations and the system of two octahedron relations have to be seen on an equal footing from the algebraical point of view. This gives a new insight into the nature of consistency (integrability) of the system of corner equations and simultaneously poses a number of open questions.

Since corner equations are elementary building blocks of Euler-Lagrange equations of a pluri-Lagrangian problem (see [BPS14]), it is quite natural to inquire about the variational structure of the corresponding system of octahedron relations. This will be the subject of our ongoing research.

Furthermore, some of the octahedron relations are known to be integrable 3D equations themselves in the sense of multidimensional consistency (see [ABS12]). For instance, relation $\Omega_2 = 0$ in the case coming from Q_1^0 , i.e., equation (15), is the fundamental equation (χ_2) in the classification in [ABS12]. Similarly, relation $\Omega_1 = 0$ in the case coming from H_1 is the equation (χ_3) in this classification. It is an open problem whether they admit a variational structure. On the other hand, the majority of octahedron relations found in the present paper do not appear in the classification in [ABS12], i.e., they are not integrable themselves in the sense of multi-dimensional consistency. It is not yet clear how they fit in the picture of integrability. Also this problem will be addressed in our future research.

There are two other octahedron type relations, i.e., relations of the type

$$R(x_1, x_2, x_3, x_{12}, x_{23}, x_{13}) = 0,$$

satisfied by solutions of corner equations. One of them is the closure relation of the corresponding discrete two-form \mathcal{L} (see [LN09, BS10, BPS14]). The second one can be expressed in terms of the biquadratic polynomials associated with the quad-equations, and reads

$$\frac{Q_{12}^{1,12}}{Q_{12}^{2,12}} \cdot \frac{Q_{23}^{2,23}}{Q_{23}^{3,23}} \cdot \frac{Q_{13}^{3,13}}{Q_{13}^{1,13}} = -1.$$

A. ABS list of quad-equations

(To prove the latter one for solutions of quad-equations, one uses the following identities:

$$\frac{Q_{12}^{0,1}}{Q_{12}^{0,2}} = \frac{Q_{12}^{1,12}}{Q_{12}^{2,12}}, \quad \frac{Q_{23}^{0,2}}{Q_{23}^{0,3}} = \frac{Q_{23}^{2,23}}{Q_{23}^{3,23}}, \quad \text{and} \quad \frac{Q_{13}^{0,3}}{Q_{13}^{0,1}} = \frac{Q_{13}^{3,13}}{Q_{13}^{1,13}},$$

which is [ABS03, formula (60)], and

$$\frac{Q_{12}^{0,1}}{Q_{12}^{0,2}} \cdot \frac{Q_{23}^{0,2}}{Q_{23}^{0,3}} \cdot \frac{Q_{13}^{0,3}}{Q_{13}^{0,1}} = -1,$$

which is [ABS03, formula (16)]. Then one shows that the resulting relation holds true for solutions of corner equations, as well.)

However, both of them can not be written as $R = 0$ with a multi-affine polynomial R , and therefore they do not qualify as octahedron relations in our sense.

Acknowledgment

This research is supported by the DFG Collaborative Research Center TRR 109 “Discretization in Geometry and Dynamics”.

A. ABS list of quad-equations

In this section we give the list of polynomials Q_{12} from the systems of quad-equations (4) for all cases we consider in this paper:

$$Q_1^\delta: Q_{12} = \alpha_1(xx_1 + x_2x_{12}) - \alpha_2(xx_2 + x_1x_{12}) - (\alpha_1 - \alpha_2)(xx_{12} + x_1x_2) + \delta\alpha_1\alpha_2(\alpha_1 - \alpha_2)$$

$$Q_2: Q_{12} = \alpha_1(xx_1 + x_2x_{12}) - \alpha_2(xx_2 + x_1x_{12}) - (\alpha_1 - \alpha_2)(xx_{12} + x_1x_2) + \alpha_1\alpha_2(\alpha_1 - \alpha_2)(x + x_1 + x_2 + x_{12}) - \alpha_1\alpha_2(\alpha_1 - \alpha_2)(\alpha_1^2 - \alpha_1\alpha_2 + \alpha_2^2)$$

$$Q_3^\delta: Q_{12} = \sin(\alpha_1)(xx_1 + x_2x_{12}) - \sin(\alpha_2)(xx_2 + x_1x_{12}) - \sin(\alpha_1 - \alpha_2)(xx_{12} + x_1x_2) + \delta \sin(\alpha_1) \sin(\alpha_2) \sin(\alpha_1 - \alpha_2)$$

$$Q_4: Q_{12} = \operatorname{sn}(\alpha_1)(xx_1 + x_2x_{12}) - \operatorname{sn}(\alpha_2)(xx_2 + x_1x_{12}) - \operatorname{sn}(\alpha_1 - \alpha_2)(xx_{12} + x_1x_2) + \operatorname{sn}(\alpha_1) \operatorname{sn}(\alpha_2) \operatorname{sn}(\alpha_1 - \alpha_2) (1 + k^2xx_1x_2x_{12})$$

$$H_1: Q_{12} = (x - x_{12})(x_1 - x_2) - \alpha_1 + \alpha_2$$

$$H_2: Q_{13} = (x - x_{12})(x_1 - x_2) - (\alpha_1 - \alpha_2)(x + x_1 + x_2 + x_{12}) - \alpha_1^2 + \alpha_2^2$$

$$H_3^\delta: Q_{12} = e^{\alpha_1}(xx_1 + x_2x_{12}) - e^{\alpha_2}(xx_2 + x_1x_{12}) + \delta(e^{2\alpha_1} - e^{2\alpha_2})$$

$$A_1^\delta: Q_{12} = \alpha_1(xx_1 + x_2x_{12}) - \alpha_2(xx_2 + x_1x_{12}) + (\alpha_1 - \alpha_2)(xx_{12} + x_1x_2) - \delta\alpha_1\alpha_2(\alpha_1 - \alpha_2)$$

$$A_2: Q_{12} = \sin(\alpha_1)(xx_2 + x_1x_{12}) - \sin(\alpha_2)(xx_1 + x_2x_{12}) - \sin(\alpha_1 - \alpha_2)(1 + xx_1x_2x_{12})$$

Here, $\delta \in \{0, 1\}$ and k is the modulus of $\operatorname{sn}(y) = \operatorname{sn}(y, k)$.

B. List of octahedron relations

In this section we give a list of the polynomials Ω_1 and Ω_2 except in the cases coming from Q_4 and A_2 . In those cases we give the polynomials R_1 . The polynomials R_2 and R_3 can be obtained from R_1 by cyclic permutations (123).

$$\begin{aligned}
 Q_1^\delta: \Omega_1 &= \alpha_1(x_2x_{12} - x_3x_{13}) + \alpha_2(x_3x_{23} - x_1x_{12}) + \alpha_3(x_1x_{13} - x_2x_{23}) \\
 &\quad - \alpha_1x_1(x_2 - x_3) - \alpha_2x_2(x_3 - x_1) - \alpha_3x_3(x_1 - x_2) \\
 &\quad + \alpha_1x_{23}(x_{13} - x_{12}) + \alpha_2x_{13}(x_{12} - x_{23}) + \alpha_3x_{12}(x_{23} - x_{13}) \\
 \Omega_2 &= x_1x_2(x_{23} - x_{13}) + x_2x_3(x_{13} - x_{12}) + x_3x_1(x_{12} - x_{23}) \\
 &\quad + x_{23}x_{13}(x_1 - x_2) + x_{13}x_{12}(x_2 - x_3) + x_{12}x_{23}(x_3 - x_1) \\
 &\quad - \delta(\alpha_1(\alpha_2 - \alpha_3)(x_1 + x_{23}) + \alpha_2(\alpha_3 - \alpha_1)(x_2 + x_{13}) + \alpha_3(\alpha_1 - \alpha_2)(x_3 + x_{12})) \\
 Q_2: \Omega_1 &= \alpha_1(x_2x_{12} - x_3x_{13}) + \alpha_2(x_3x_{23} - x_1x_{12}) + \alpha_3(x_1x_{13} - x_2x_{23}) \\
 &\quad - \alpha_1x_1(x_2 - x_3) - \alpha_2x_2(x_3 - x_1) - \alpha_3x_3(x_1 - x_2) \\
 &\quad + \alpha_1x_{23}(x_{13} - x_{12}) + \alpha_2x_{13}(x_{12} - x_{23}) + \alpha_3x_{12}(x_{23} - x_{13}) \\
 &\quad + \alpha_1^2(\alpha_2 - \alpha_3)(x_1 + x_2 - x_{23} - x_{13}) + \alpha_2^2(\alpha_3 - \alpha_1)(x_2 + x_3 - x_{13} - x_{12}) \\
 &\quad + \alpha_3^2(\alpha_1 - \alpha_2)(x_3 + x_1 - x_{12} - x_{23}) \\
 \Omega_2 &= x_1x_2(x_{23} - x_{13}) + x_2x_3(x_{13} - x_{12}) + x_3x_1(x_{12} - x_{23}) \\
 &\quad + x_{23}x_{13}(x_1 - x_2) + x_{13}x_{12}(x_2 - x_3) + x_{12}x_{23}(x_3 - x_1) \\
 &\quad - 2\alpha_1(\alpha_2 - \alpha_3)x_1x_{23} - 2\alpha_2(\alpha_3 - \alpha_1)x_2x_{13} - 2\alpha_3(\alpha_1 - \alpha_2)x_3x_{12} \\
 &\quad + \alpha_1(\alpha_2 - \alpha_3)(x_2x_3 + x_{13}x_{12}) + \alpha_2(\alpha_3 - \alpha_1)(x_3x_1 + x_{12}x_{23}) \\
 &\quad + \alpha_3(\alpha_1 - \alpha_2)(x_1x_2 + x_{23}x_{13}) + \alpha_1(\alpha_2 - \alpha_3)(x_2x_{12} + x_3x_{13}) \\
 &\quad + \alpha_2(\alpha_3 - \alpha_1)(x_3x_{23} + x_1x_{12}) + \alpha_3(\alpha_1 - \alpha_2)(x_1x_{13} + x_2x_{23}) \\
 &\quad + \alpha_1(\alpha_2 - \alpha_3)(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 - \alpha_1(\alpha_2 + \alpha_3))(x_1 + x_{23}) \\
 &\quad + \alpha_2(\alpha_3 - \alpha_1)(\alpha_2^2 + \alpha_3^2 + \alpha_1^2 - \alpha_2(\alpha_3 + \alpha_1))(x_2 + x_{13}) \\
 &\quad + \alpha_3(\alpha_1 - \alpha_2)(\alpha_3^2 + \alpha_1^2 + \alpha_2^2 - \alpha_3(\alpha_1 + \alpha_2))(x_3 + x_{12}) \\
 &\quad + 2\alpha_1\alpha_2\alpha_3(\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1) \\
 Q_3^\delta: \Omega_1 &= \sin(\alpha_1)(x_2x_{12} - x_3x_{13}) + \sin(\alpha_2)(x_3x_{23} - x_1x_{12}) + \sin(\alpha_3)(x_1x_{13} - x_2x_{23}) \\
 &\quad - \sin(\alpha_1 - \alpha_2)(x_1x_2 - x_{23}x_{13}) - \sin(\alpha_2 - \alpha_3)(x_2x_3 - x_{13}x_{12}) \\
 &\quad - \sin(\alpha_3 - \alpha_1)(x_3x_1 - x_{12}x_{23}) \\
 \Omega_2 &= \cos(\frac{1}{2}(\alpha_1 - \alpha_2 - \alpha_3))(x_1 + x_{23})(x_2x_{13} - x_3x_{12}) \\
 &\quad + \cos(\frac{1}{2}(\alpha_2 - \alpha_3 - \alpha_1))(x_2 + x_{13})(x_3x_{12} - x_1x_{23}) \\
 &\quad + \cos(\frac{1}{2}(\alpha_3 - \alpha_1 - \alpha_2))(x_3 + x_{12})(x_1x_{23} - x_2x_{13}) \\
 &\quad + 4\delta \sin(\alpha_1) \sin(\alpha_2 - \alpha_3) \cos(\frac{1}{2}(\alpha_1 - \alpha_2 - \alpha_3))(x_1 + x_{23}) \\
 &\quad + 4\delta \sin(\alpha_2) \sin(\alpha_3 - \alpha_1) \cos(\frac{1}{2}(\alpha_2 - \alpha_3 - \alpha_1))(x_2 + x_{13}) \\
 &\quad + 4\delta \sin(\alpha_3) \sin(\alpha_1 - \alpha_2) \cos(\frac{1}{2}(\alpha_3 - \alpha_1 - \alpha_2))(x_3 + x_{12}) \\
 Q_4: R_1 &= \text{sn}(\alpha_1)\tilde{R}_1 \text{ with } \tilde{R}_1 = \text{sn}(\alpha_3)(x_1x_{13} - x_2x_{23}) + \text{sn}(\alpha_2)(x_3x_{23} - x_1x_{12}) \\
 &\quad + \text{sn}(\alpha_1)(1 + k^2 \text{sn}(\alpha_3) \text{sn}(\alpha_2) \text{sn}(\alpha_3 - \alpha_1) \text{sn}(\alpha_1 - \alpha_2))(x_2x_{12} - x_3x_{13}) \\
 &\quad - \text{sn}(\alpha_3 - \alpha_1)(x_3x_1 - x_{12}x_{23}) - \text{sn}(\alpha_1 - \alpha_2)(x_1x_2 - x_{23}x_{13}) \\
 &\quad - \text{sn}(\alpha_2 - \alpha_3)(1 + k^2 \text{sn}(\alpha_3) \text{sn}(\alpha_2) \text{sn}(\alpha_3 - \alpha_1) \text{sn}(\alpha_1 - \alpha_2))(x_2x_3 - x_{13}x_{12}) \\
 &\quad + k^2 \text{sn}(\alpha_3) \text{sn}(\alpha_2) \text{sn}(\alpha_1 - \alpha_2)(x_1x_2x_{13}x_{12} - x_2x_3x_{23}x_{13}) \\
 &\quad - k^2 \text{sn}(\alpha_3) \text{sn}(\alpha_2) \text{sn}(\alpha_3 - \alpha_1)(x_2x_3x_{12}x_{23} - x_3x_1x_{13}x_{12}) \\
 &\quad + k^2 \text{sn}(\alpha_3) \text{sn}(\alpha_3 - \alpha_1) \text{sn}(\alpha_1 - \alpha_2)(x_3x_1x_2x_{13} - x_2x_{13}x_{12}x_{23}) \\
 &\quad - k^2 \text{sn}(\alpha_2) \text{sn}(\alpha_3 - \alpha_1) \text{sn}(\alpha_1 - \alpha_2)(x_1x_2x_3x_{12} - x_3x_{12}x_{23}x_{13})
 \end{aligned}$$

References

$$\begin{aligned}
H_1: \quad \Omega_1 &= x_1(x_{13} - x_{12}) + x_2(x_{12} - x_{23}) + x_3(x_{23} - x_{13}) \\
\Omega_2 &= x_1x_2(x_{23} - x_{13}) + x_2x_3(x_{13} - x_{12}) + x_3x_1(x_{12} - x_{23}) \\
&\quad + x_{23}x_{13}(x_1 - x_2) + x_{13}x_{12}(x_2 - x_3) + x_{12}x_{23}(x_3 - x_1) \\
&\quad + (\alpha_1 - \alpha_2)(x_3 + x_{12}) + (\alpha_2 - \alpha_3)(x_1 + x_{23}) + (\alpha_3 - \alpha_1)(x_2 + x_{13}) \\
H_2: \quad \Omega_1 &= x_1(x_{13} - x_{12}) + x_2(x_{12} - x_{23}) + x_3(x_{23} - x_{13}) \\
&\quad - \alpha_1(x_2 - x_3 + x_{13} - x_{12}) - \alpha_2(x_3 - x_1 + x_{12} - x_{23}) - \alpha_3(x_1 - x_2 + x_{23} - x_{13}) \\
\Omega_2 &= x_1x_2(x_{23} - x_{13}) + x_2x_3(x_{13} - x_{12}) + x_3x_1(x_{12} - x_{23}) \\
&\quad + x_{23}x_{13}(x_1 - x_2) + x_{13}x_{12}(x_2 - x_3) + x_{12}x_{23}(x_3 - x_1) \\
&\quad + 2(\alpha_1 - \alpha_2)x_3x_{12} + 2(\alpha_2 - \alpha_3)x_1x_{23} + 2(\alpha_3 - \alpha_1)x_2x_{13} \\
&\quad + (\alpha_1 - \alpha_2)(x_1x_2 + x_{23}x_{13}) + (\alpha_2 - \alpha_3)(x_2x_3 + x_{13}x_{12}) \\
&\quad + (\alpha_3 - \alpha_1)(x_3x_1 + x_{12}x_{23}) \\
&\quad + (\alpha_1^2 - \alpha_2^2)(x_3 + x_{12}) + (\alpha_2^2 - \alpha_3^2)(x_1 + x_{23}) + (\alpha_3^2 - \alpha_1^2)(x_2 + x_{13}) \\
&\quad - 2(\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1) \\
H_3^\delta: \quad \Omega_1 &= e^{\alpha_3}x_1(x_{13} - x_{12}) + e^{\alpha_1}x_2(x_{12} - x_{23}) + e^{\alpha_2}x_3(x_{23} - x_{13}) \\
\Omega_2 &= x_1x_2(e^{\alpha_3+\alpha_1}x_{23} - e^{\alpha_2+\alpha_3}x_{13}) + x_2x_3(e^{\alpha_1+\alpha_2}x_{13} - e^{\alpha_3+\alpha_1}x_{12}) \\
&\quad + x_3x_1(e^{\alpha_2+\alpha_3}x_{12} - e^{\alpha_1+\alpha_2}x_{23}) + x_{23}x_{13}(e^{\alpha_3+\alpha_1}x_1 - e^{\alpha_2+\alpha_3}x_2) \\
&\quad + x_{13}x_{12}(e^{\alpha_1+\alpha_2}x_2 - e^{\alpha_3+\alpha_1}x_3) + x_{12}x_{23}(e^{\alpha_2+\alpha_3}x_3 - e^{\alpha_1+\alpha_2}x_1) \\
&\quad - \delta(e^{\alpha_3+2\alpha_1} - e^{\alpha_3+2\alpha_2})(x_3 + x_{12}) - \delta(e^{\alpha_1+2\alpha_2} - e^{\alpha_1+2\alpha_3})(x_1 + x_{23}) \\
&\quad - \delta(e^{\alpha_2+2\alpha_3} - e^{\alpha_2+2\alpha_1})(x_2 + x_{13}) \\
A_1^\delta: \quad \Omega_1 &= \alpha_1(x_2x_{12} - x_3x_{13}) + \alpha_2(x_3x_{23} - x_1x_{12}) + \alpha_3(x_1x_{13} - x_2x_{23}) \\
&\quad + \alpha_1x_1(x_2 - x_3) + \alpha_2x_2(x_3 - x_1) + \alpha_3x_3(x_1 - x_2) \\
&\quad - \alpha_1x_{23}(x_{13} - x_{12}) - \alpha_2x_{13}(x_{12} - x_{23}) - \alpha_3x_{12}(x_{23} - x_{13}) \\
\Omega_2 &= x_1x_2((\alpha_1 - \alpha_2 + \alpha_3)x_{23} - (\alpha_3 - \alpha_1 + \alpha_2)x_{13}) \\
&\quad + x_2x_3((\alpha_2 - \alpha_3 + \alpha_1)x_{13} - (\alpha_1 - \alpha_2 + \alpha_3)x_{12}) \\
&\quad + x_3x_1((\alpha_3 - \alpha_1 + \alpha_2)x_{12} - (\alpha_2 - \alpha_3 + \alpha_1)x_{23}) \\
&\quad + x_{23}x_{13}((\alpha_1 - \alpha_2 + \alpha_3)x_1 - (\alpha_3 - \alpha_1 + \alpha_2)x_2) \\
&\quad + x_{13}x_{12}((\alpha_2 - \alpha_3 + \alpha_1)x_2 - (\alpha_1 - \alpha_2 + \alpha_3)x_3) \\
&\quad + x_{12}x_{23}((\alpha_3 - \alpha_1 + \alpha_2)x_3 - (\alpha_2 - \alpha_3 + \alpha_1)x_1) \\
&\quad - \delta\alpha_1(\alpha_2 - \alpha_3)(\alpha_1 - \alpha_2 - \alpha_3)(x_1 + x_{23}) \\
&\quad - \delta\alpha_2(\alpha_3 - \alpha_1)(\alpha_2 - \alpha_3 - \alpha_1)(x_2 + x_{13}) \\
&\quad - \delta\alpha_3(\alpha_1 - \alpha_2)(\alpha_3 - \alpha_1 - \alpha_2)(x_3 + x_{12}) \\
A_2: \quad R_1 &= \sin(\alpha_3)\sin(\alpha_1)(x_3x_{23} - x_1x_{12}) + \sin(\alpha_1)\sin(\alpha_2)(x_1x_{13} - x_2x_{23}) \\
&\quad + \sin(\alpha_2)\sin(\alpha_3)(x_2x_{12} - x_3x_{13}) + \sin(\alpha_3 - \alpha_1)\sin(\alpha_1 - \alpha_2)(x_3x_{13} - x_2x_{12}) \\
&\quad + \sin(\alpha_1)\sin(\alpha_3 - \alpha_1)(x_3x_1x_{13}x_{12} - x_2x_3x_{12}x_{23}) \\
&\quad + \sin(\alpha_1)\sin(\alpha_1 - \alpha_2)(x_1x_2x_{13}x_{12} - x_2x_3x_{23}x_{13})
\end{aligned}$$

References

- [ABS03] Vsevolod E. Adler, Alexander I. Bobenko, and Yuri B. Suris, *Classification of Integrable Equations on Quad-Graphs. The Consistency Approach*, Comm. Math. Phys. **233** (2003), pp. 513–543.
- [ABS12] ———, *Classification of integrable discrete equations of octahedron type*, Intern. Math. Research Notices **2012** (2012), no. 8, pp. 1822–1889.
- [Bou72] Nicolas Bourbaki, *Commutative algebra*, Elements of Mathematics, Hermann, Paris, 1972.

References

- [BPS14] Raphael Boll, Matteo Petrera, and Yuri B. Suris, *What is integrability of discrete variational systems?*, Proc. R. Soc. A **470** (2014), no. 20130550.
- [BS02] Alexander I. Bobenko and Yuri B. Suris, *Integrable systems on quad-graphs*, Intern. Math. Research Notices **2002** (2002), 573–611.
- [BS10] ———, *On the Lagrangian structure of integrable quad-equations*, Lett. Math. Phys. **92** (2010), no. 1, pp. 17–31.
- [LN09] Sarah Lobb and Frank W. Nijhoff, *Lagrangian multiforms and multidimensional consistency*, J. Phys. A: Math. Theor. **42** (2009), no. 454013.
- [Nij02] Frank W. Nijhoff, *Lax pair for the Adler (lattice Krichever-Novikov) system*, Phys. Lett. A **297** (2002), pp. 49–58.